

CCP Maths 2

Reduction , projector, endomorphism of rank 1 ...
Hadamard's inequality and some applications

Solution

Exercise 1.

1. 1.1 A is a symmetric matrix, so diagonalizable.

1.2 Diagonalization of A : A characteristic polynomial of A is $\chi_A = -(X+4)(X-1)(X-6)$, so $\text{Sp}(A) = \{-4, 1, 6\}$.

$$A = PDP^{-1}, \text{ where } P = \begin{pmatrix} 3 & -4 & 3 \\ -5 & 0 & 5 \\ 4 & 3 & 4 \end{pmatrix} \text{ and } D = \text{diag}(-4, 1, 6).$$

1.3 With *wxMaxima* :

$$A^n = PD^nP^{-1} = \begin{pmatrix} \frac{9}{50}(-4)^n + \frac{9}{50}6^n + \frac{16}{25} & -\frac{3}{10}(-4)^n + \frac{3}{10}6^n & \frac{6}{25}(-4)^n + \frac{6}{25}6^n - \frac{12}{25} \\ -\frac{3}{10}(-4)^n + \frac{3}{10}6^n & \frac{1}{2}(-4)^n + \frac{1}{2}6^n & -\frac{2}{5}(-4)^n + \frac{2}{5}6^n + \frac{16}{25} \\ \frac{6}{25}(-4)^n + \frac{6}{25}6^n - \frac{12}{25} & -\frac{2}{5}(-4)^n + \frac{2}{5}6^n & \frac{8}{25}(-4)^n + \frac{8}{25}6^n + \frac{9}{25} \end{pmatrix}$$

2. Expression of sequences u_n, v_n and w_n : With *wxMaxima*

For $n \in \mathbb{N}$, we put $U_n = \begin{pmatrix} u_n \\ v_n \\ w_n \end{pmatrix}$, the recurrence relation between the sequences, $(u_n)_n, (v_n)_n$ and $(w_n)_n$ can be written matrixially; $U_{n+1} = AU_n$, so for each $n \in \mathbb{N}$; $U_n = A^n U_0$, we get :

$$\begin{cases} u_n &= \frac{21}{50}(-4)^n + \frac{21}{50}6^n + \frac{4}{25} \\ v_n &= -\frac{7}{10}(-4)^n + \frac{7}{10}6^n \\ w_n &= \frac{14}{25}(-4)^n + \frac{14}{25}6^n - \frac{3}{25} \end{cases}$$

Exercise 2.

1. 1.1 p is a projector of E , so the polynomial $X^2 - X = X(X-1)$ is an annihilator of p , since the two polynomials X and $X-1$ are coprime, by the primary decomposition theorem, we have; $E = \ker p \oplus \ker(p - \text{Id}_E)$; other hand $x \in \text{Im } p$ if and only if $x = p(x)$. This show that $\ker(p - \text{Id}_E) = \text{Im } p$. Whence $E = \ker p \oplus \text{Im } p$.

1.2 We denote $r = \text{rg } p = \dim(\text{Im } p)$, (v_1, \dots, v_r) a basis of a subspace $\text{Im } p$ and (v_{r+1}, \dots, v_n) a basis of a subspace $\ker p$; it is clear that (v_1, \dots, v_n) is a basis of E , the matrix of p in this basis has the form : $\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, it follows that $\text{tr } p = r = \text{rg } p$.

1.3 Let u be the endomorphism of \mathbb{R}^2 canonically associated to the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we have $\text{rg } u = 2 = \text{tr } u$, but u is not a projector, since $A^2 \neq A$.

Exercice 2. suite

2. We consider a matrices $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ we see that $\text{rg} A = \text{rg} B = 1$.

A matrix A is diagonal, so, it is diagonalizable.

A matrix B is not diagonalizable : B is nilpotent, and it's minimal polynomial is X^2 , it is a split polynomial but it has a double root.

3. u an endomorphism of E of rank 1.

3.1 By the rank formula $\dim \ker u = n - \text{rg} u = n - 1$, so $\ker u$ is an hyperplane of E . Now let (e_1, \dots, e_{n-1}) be a basis of $\ker u$ there exist $e_n \in E$ such that $(e_1, \dots, e_{n-1}, e_n)$ be a basis of E , obviously $e_n \notin \ker u$, we can write $u(e_n) = a_1 e_1 + \dots + a_n e_n$, for all $i \in \{1, \dots, n-1\}$, we have $u(e_i) = 0$. Thus the matrix of u with respect to $B = (e_1, \dots, e_n)$ has desired form.

3.2 With the notations of previous question, we have $\text{tr} u = a_n$, and the characteristic polynomial of u is $\chi_u = X^{n-1}(X - a_n)$.

If $\text{tr} u \neq 0$ i.e $a_n \neq 0$, in this case $\text{Sp} u = \{0, a_n\}$, the eigenspace associated to the eigenvalue 0 is $\ker u$ and $\dim \ker u = n - 1$, other $a_n = \text{tr} u$ is an eigenvalue of u then $\dim \ker(u - \text{Id}_E) \geq 1$ as $\ker u \cap \ker(u - a_n \text{Id}_E) = \{0\}$ it result that $\dim \ker(u - a_n \text{Id}_E) = 1$, that is $n = \dim \ker u + \dim \ker(u - a_n \text{Id}_E)$, we deduce that u est diagonalizable.

If $\text{tr} u = 0$ i.e $a_n = 0$, then 0 is the unique eigenvalue of u , hence u is a nilpotent endomorphism, but the unique nilpotent endomorphism which is diagonalizable is the zero endomorphism, since u is not a zero endomorphism, it follows that u is not diagonalizable.

Remark: If f is a nilpotent endomorphism of vector space of dimension n that is diagonalizable, then $f = 0$; In fact, let \mathcal{B} be a basis consisting of eigenvectors of the endomorphism f , the matrix of f with respect to \mathcal{B} is a zero matrix, hence f is a zero endomorphism.

3.3 $\text{rg} u = 1$ et $\text{tr} u = 1 \neq 0$, by the result of previous question, u is diagonalizable ($\dim \ker u = n - 1$ and $\dim \ker(u - \text{Id}_E) = 1$) and there exist a basis B consisting of eigenvectors of u such that $A :=$

$$\mathcal{M}_{\mathcal{B}}(u) = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{pmatrix}, \text{ it is clear that } A^2 = A, \text{ that is } u^2 = u \text{ hence } u \text{ is a projector.}$$

3.4 Let u be the endomorphism of \mathbb{R}^3 canonically associated to matrix A , it is easy to verify that $\text{tr} A = 1$ $\text{rg} A = 1$, so $\text{tr} u = \text{rg} u = 1$; by the previous result, u is a projector.

If we denote (e_1, e_2, e_3) the canonical basis of \mathbb{R}^3 ; then $u(e_1) = u(e_2) = -u(e_3) = e_1 + e_2 + e_3$, hence $\text{Im} u = \text{Vect}(e_1 + e_2 + e_3)$.

Let $(x, y, z) \in \mathbb{R}^3$, then $u(x, y, z) = 0$ if and only if $x + y - z = 0$ if and only if $z = x + y$, it follows that $\ker u = \text{Vect}((1, 0, 1), (0, 1, 1))$.

PROBLEM

First part : Preliminary questions

1. 1.1 Any symmetric endomorphism is diagonalizable in an orthonormal basis.
Any symmetric matrix is orthogonally diagonalizable.

1.2 The matrix S is symmetric, its characteristic polynomial is X^2 , since S is not a zero matrix, then its minimal polynomial is X^2 that is split and it has a double root, so S is not diagonalizable.
Conclusion : A complex symmetric matrix is not in general diagonalizable .

2. 2.1 Let $x \in E$, by putting $x = \sum_{i=1}^n x_i \varepsilon_i$, we have $s(x) = \sum_{i=1}^n \lambda_i x_i \varepsilon_i$, hence $\langle s(x), x \rangle = \sum_{i=1}^n \lambda_i x_i^2$.

2.2 Let $x = \sum_{i=1}^n x_i \varepsilon_i \in S(0, 1)$, then $\|x\|^2 = \sum_{i=1}^n x_i^2$ (since β is an orthonormal basis of E),

$$\lambda_1 = \lambda_1 \sum_{i=1}^n x_i^2 \leq R_s(x) = \sum_{i=1}^n \lambda_i x_i^2 \leq \lambda_n \sum_{i=1}^n x_i^2 = \lambda_n$$

thus $R_s(x) \in [\lambda_1, \lambda_n]$.

3. 3.1 Let λ be an eigenvalue of s , and let v be an associated eigenvector, that is $s(v) = \lambda v$, we have $\langle s(v), v \rangle = \lambda \|v\|^2$, if s is positive then $\lambda \|v\|^2 \geq 0$, and $\lambda \geq 0$. If s is positive definite then $\lambda \|v\|^2 > 0$, and $\lambda > 0$.

3.2 $\langle s(e_i), e_j \rangle = s_{i,j}$, in particular $s_{i,i} = \langle s(e_i), e_i \rangle = R_s(e_i) \in [\lambda_1, \lambda_n]$, thus $\lambda_1 \leq s_{i,i} \leq \lambda_n$.

Remark: If $x = \sum_{i=1}^n x_i \varepsilon_i$ and $y = \sum_{i=1}^n y_i \varepsilon_i$ then $\langle s(x), y \rangle = \sum_{1 \leq i, j \leq n} x_i y_j s_{i,j}$.

4. The maps $M \mapsto M$ and $M \mapsto {}^t M$ are continuous, (linear in finite dimension), so $M \mapsto {}^t M M$ continuous (product of continuous maps), thus $M \mapsto {}^t M M - I_n$ is a continuous map.

5. Let $A \in \mathcal{O}_n(\mathbb{R})$, then these row vectors form an orthonormal basis of \mathbb{R}^n , in particular each row is an unitary vector. Now for all $i \in \{1, \dots, n\}$ we have $\sum_{k=1}^n a_{i,k}^2 = 1$, thus for all $j \in \{1, \dots, n\}$, $a_{i,j}^2 \leq \sum_{k=1}^n a_{i,k}^2 = 1$, so $|a_{i,j}| \leq 1$.

6. We endow $\mathcal{M}_n(\mathbb{R})$ with the norm N defined for any $M = (m_{i,j})$ by $N(M) = \max_{1 \leq i, j \leq n} |a_{i,j}|$, and we recall that ; in finite dimension, all norms are equivalent .

The map $f : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R})$ defined by $f(M) = {}^t M M - I_n$ is continuous, and $\mathcal{O}_n(\mathbb{R}) = f^{-1}\{0\}$, so $\mathcal{O}_n(\mathbb{R})$ closed subset of $\mathcal{M}_n(\mathbb{R})$, other hand, for any $M \in \mathcal{O}_n(\mathbb{R})$, we have $N(M) \leq 1$, hence $\mathcal{O}_n(\mathbb{R})$ is a bounded subset, since $\mathcal{M}_n(\mathbb{R})$ is finite dimensional vector space, the subset $\mathcal{O}_n(\mathbb{R})$ is a compact subset of $\mathcal{M}_n(\mathbb{R})$.

7. 7.1 Since S is symmetric, there exist orthogonal matrix P such that $S = {}^t P \Delta P$, $T(A) = \text{tr}(AS) = \text{tr}(A {}^t P \Delta P) = \text{tr}(P A {}^t P \Delta)$, and $B = P A {}^t P \in \mathcal{O}_n(\mathbb{R})$ is a suitable matrix.

7.2 The map $M \mapsto \text{tr}(MS)$ is a linear form of $\mathcal{M}_n(\mathbb{R})$ so it is a continuous map, thus its restriction T to $\mathcal{O}_n(\mathbb{R})$ is a continuous map, since $\mathcal{O}_n(\mathbb{R})$ is a compact subset, then T is bounded and attain its upper bound its maximum .

7.3 Let $A \in \mathcal{O}_n(\mathbb{R})$, there exist $B = (b_{i,j}) \in \mathcal{O}_n(\mathbb{R})$ such that $T(A) = \text{tr}(B \Delta)$, for all $i \in \{1, \dots, n\}$, we have $(B \Delta)_{i,i} = \lambda_i b_{i,i}$, hence $\text{tr}(B \Delta) = \sum_{i=1}^n \lambda_i b_{i,i}$, since $B \in \mathcal{O}_n(\mathbb{R})$, then $|b_{i,i}| \leq 1$. Now it is that

$$T(A) = \text{tr}(B \Delta) \leq \sum_{i=1}^n \lambda_i = \text{tr}(S) \text{ (the } \lambda_i \geq 0 \text{)}.$$

We have $I_n \in \mathcal{O}_n(\mathbb{R})$ et $T(I_n) = \text{tr}(S)$, by the previous question, for all $A \in \mathcal{O}_n(\mathbb{R})$, we get $T(A) \leq \text{tr}(S) = T(I_n)$, hence $t = T(I_n) = \text{tr}(S)$.

Second part :
Hadamard's inequality

8. $\det S = \prod_{i=1}^n \lambda_i$ et $\text{tr}(S) = \sum_{i=1}^n \lambda_i$, by the arithmetic-geometric inequality, we get $\prod_{i=1}^n \lambda_i \leq \left(\frac{1}{n} \sum_{i=1}^n \lambda_i\right)^n$, i.e $\det S \leq \left(\frac{1}{n} \text{tr}(S)\right)^n$.
9. ${}^t S_\alpha = {}^t(DSD) = {}^t D^t S D = {}^t D S D = S_\alpha$, hence S_α is a symmetric matrix. For any column matrix X , ${}^t X S_\alpha X = {}^t(DX)S(DX) \geq 0$, then $S_\alpha \in \mathcal{S}_n^+(\mathbb{R})$.
By remarking that ${}^t D = D$, $\text{tr}(S_\alpha) = \text{tr}(SD^2)$, and for any $i \in \{1, \dots, n\}$, $(SD^2)_{i,i} = s_{i,i} \alpha_i^2$, it follow that $\text{tr}(S_\alpha) = \sum_{i=1}^n s_{i,i} \alpha_i^2$.
10. $S_\alpha \in \mathcal{S}_n^+(\mathbb{R})$, by the result of question 8., we get $\det S_\alpha \leq \left(\frac{1}{n} \text{tr}(S_\alpha)\right)^n$, but $S_\alpha = \det S \det D^2 = \prod_{i=1}^n \lambda_i \prod_{i=1}^n \frac{1}{s_{i,i}} = \frac{\prod_{i=1}^n \lambda_i}{\prod_{i=1}^n s_{i,i}}$, and $\text{tr}(S_\alpha) = \sum_{i=1}^n s_{i,i} \alpha_i^2 = n$, so $\det S = \prod_{i=1}^n \lambda_i \leq \prod_{i=1}^n s_{i,i}$.
11. S_ϵ is positive symmetric matrix. Moreover, for all $i \in \{1, \dots, n\}$, $(S_\epsilon)_{i,i} = s_{i,i} + \epsilon > 0$, by the result of the 10., it follows that $\det S_\epsilon \leq \prod_{i=1}^n (S_\epsilon)_{i,i} = \prod_{i=1}^n (s_{i,i} + \epsilon)$.
passing to the limit as ϵ tends to 0 in the previous inequality, and taking account the continuity of the maps \det and tr , we obtain $\det S \leq \prod_{i=1}^n s_{i,i}$, this show the result.

Third part :

Application of Hadamard's inequality : determining a minimum

12. The matrix B is symmetric, and for any column vector X we have ${}^t X B X = {}^t(\Omega X) A (\Omega X) \geq 0$. If ${}^t X A X = 0$ then ${}^t(\Omega X) A (\Omega X) = 0$ as A positive definite, then $\Omega X = 0$, hence $X = 0$ (Ω is invertible matrix), other hand $\det B = \det A = 1$, thus $B \in \mathcal{U}$.
 $\text{tr}(AS) = \text{tr}(A\Omega\Delta^t\Omega) = \text{tr}({}^t\Omega A \Omega\Delta) = \text{tr}(B\Delta)$.
13. If $A \in \mathcal{U}$, by putting $B = {}^t\Omega A \Omega$, we get $B \in \mathcal{U}$ and $\text{tr}(AS) = \text{tr}(B\Delta)$.
If $B \in \mathcal{U}$, by putting $A = \Omega B^t \Omega$, we get $A \in \mathcal{U}$ et $\text{tr}(B\Delta) = \text{tr}(AS)$.
This show the equality between these sets.
If $B = (b_{i,j}) \in \mathcal{U}$, we have $\text{tr}(B\Delta) = \sum_{i=1}^n \lambda_i b_{i,i}$, taking account the positivity of $b_{i,i}$ and λ_i , we obtain $\text{tr}(B\Delta) \geq 0$. It follows that the subset $\{\text{tr}(B\Delta), B \in \mathcal{U}\}$ is non-empty set and it is bounded below, thus it has a lower bounded.
14. Let $B = (b_{i,i}) \in \mathcal{U}$, we have $\text{tr}(B\Delta) = \sum_{i=1}^n \lambda_i b_{i,i}$, by arithmetic-geometric inequality, with $a_i = \lambda_i b_{i,i}$, we obtain $\sum_{i=1}^n \lambda_i b_{i,i} \geq n \left(\prod_{i=1}^n \lambda_i b_{i,i}\right)^{\frac{1}{n}} = n(\lambda_1 \dots \lambda_n)^{\frac{1}{n}} (b_{1,1} \dots b_{n,n})^{\frac{1}{n}}$. this show the result.
15. $B \in \mathcal{U} \subset \mathcal{S}_n^+(\mathbb{R})$, by applying the Hadamard's inequality to the matrix B , we get $b_{1,1} \dots b_{n,n} \geq \det B = 1$, hence $\text{tr}(B\Delta) \geq n(\lambda_1 \dots \lambda_n)^{\frac{1}{n}} = n(\det S)^{\frac{1}{n}}$.
16. Immediately we have : $D \in \mathcal{U}$ and $\text{tr}(D\Delta) = \sum_{i=1}^n \lambda_i \mu_i = n(\det S)^{\frac{1}{n}}$, so for all $B \in \mathcal{U}$, we have $\text{tr}(B\Delta) \geq \text{tr}(D\Delta)$. It follows that $m = \text{tr}(D\Delta) = n(\det S)^{\frac{1}{n}}$.

